Combinatorics and Graph Theory III
Tutorial 9

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1 VC-dimension

Let H be a set system and C a set. We define $H \cap C = \{h \cap C \mid h \in H\}$. C is said to be *broken* by H if $|H \cap C| = 2^{|C|}$. The *VC-dimension* of H is the largest cardinality of a set that is shattered by H (it can be infinite).

1. What is the VC-dimension of the set system of all subsets of $\mathbb{R}^2$?
2. What is the VC-dimension of the set system of all convex polygons?
3. What is the VC-dimension of the set system of all half-planes?
4. Consider the system of all axis-aligned rectangles. Prove that it does not break all sets of four points.
5. Prove that the VC-dimension of the set system of all axis-aligned rectangles is 4.

Let $\mathcal{F}$ be a finite set system. We want to prove that $\mathcal{F}$ breaks at least $|\mathcal{F}|$ subsets of $\bigcup_{F\in\mathcal{F}} F$.

6. Prove it whenever $|\mathcal{F}| \leq 1$.

Thus we can suppose $|\mathcal{F}| \geq 2$, and thus there exists $c \in \bigcup_{F\in\mathcal{F}} F$ that does not appear in every set of $\mathcal{F}$.
Let $\mathcal{F}_1$ be the sets of $\mathcal{F}$ containing c, and $\mathcal{F}_2$ those not containing c.

7. What can you say about a set broken by $\mathcal{F}_1$ or $\mathcal{F}_2$?

For $i = 1, 2$, let $\mathcal{R}_i$ be the system of subsets of $\bigcup_{F\in\mathcal{F}_i} F$ broken by $\mathcal{F}_i$, and $R_3 = \{X \cup \{c\} \mid X \in \mathcal{R}_1 \cap \mathcal{R}_2\}$.

8. What can you say about $|\mathcal{R}_3|$?
9. Find $|\mathcal{R}_1 \cup \mathcal{R}_2| + |\mathcal{R}_3|$ sets broken by $\mathcal{F}$, and conclude!
10. Let X be a finite set and $\mathcal{F}$ a set system. Prove that:

$$|X \cap \mathcal{F}| \leq \sum_{i=0}^{k} \binom{|X|}{i}$$