Combinatorics and Graph Theory III
Tutorial 8

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1 Orientation

Please remind the following theorem:

Theorem 1.1 (*Hall,1935*)

Let $G = (X \cup Y, E)$ be a bipartite graph with parts X and Y. G admits a matching covering X if and only, for every $W \subseteq X$, $|W| \leq |\{v \in Y \mid \exists w \in W, wv \in E\}|$.

1. Prove that if a graph G has an orientation in which every vertex has indegree at most d, then every subgraph H of G verifies $|E(H)| \leq d|V(H)|$.

We want to prove that the converse is true.

2. Prove that if every subgraph H of a graph G verifies $|E(H)| \leq d|V(H)|$ then G admits an orientation in which every vertex has indegree at most d.
hint: consider the bipartite graph with vertex set $(V(G) \times \{1, \dots, d\}) \cup E(G)$ with an edge $(u, i)e$ for every $i \in \{1, \dots, d\}$ and every endpoint u of every edge e.

3. Prove that every planar graph admits an orientation in which every vertex has indegree at most 3.

4. Prove that every bipartite planar graph admits an orientation in which every vertex has indegree at most 2.

2 List-colouring

Given a graph G and a list $L(v)$ of colours for each vertex $v \in V(G)$, a *list colouring* is a choice function that assigns a colour in $L(v)$ to every vertex $v \in V(G)$ so that adjacent vertices get assigned distinct colours. A graph is *k-list-colourable* if it has a list colouring for every possible choice of L with $|L(v)| = k$ for every vertex v. The *list chromatic number* of a graph G is the least number k such that G is k-list colourable.

1. Prove that the list chromatic number of odd cycles is 3.

2. Prove that the list chromatic number of even cycles is 2.

3. Prove that the list chromatic number of $K_{3,3}$ is 3.

4. Prove that the list chromatic number of bipartite graphs is unbounded.