Combinatorics and Graph Theory III
Tutorial 7

Guillaume Aubian

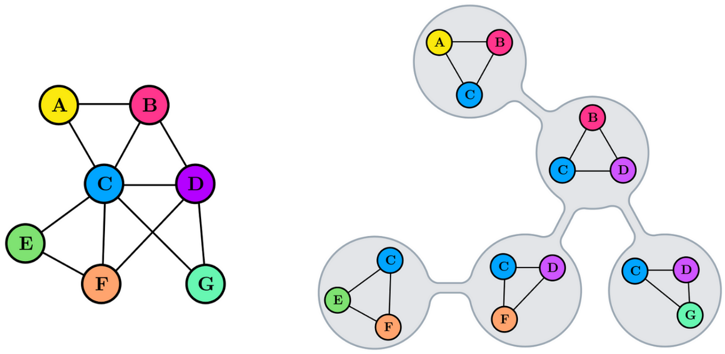
1 Treewidth and Brambles

Definition 1.1

Given a graph G, a tree decomposition *is a pair (T, β) with T a tree and $\beta : V(T) \to 2^{V(G)}$. β assigns a bag $\beta(n)$ to each vertex n of T and obeys the following rules:*

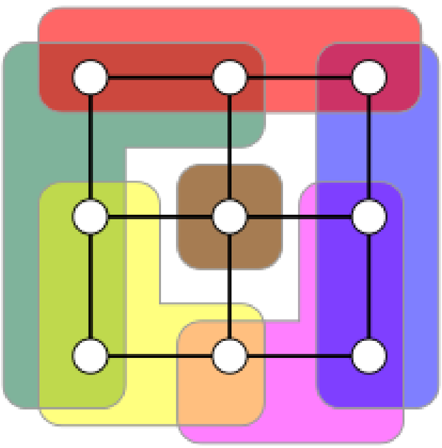
- *for every $v \in V(G)$, there exists $n \in V(T)$ such that $v \in \beta(n)$,*
- *for every $uv \in E(G)$, there exists $n \in V(T)$ such that $u, v \in \beta(n)$,*
- *for every $v \in V(G)$, the set $\{n \in V(T) \mid v \in \beta(n)\}$ induces a connected subtree of T.*

The *width* of a tree decomposition is the size of the largest bag minus one. The *treewidth* $tw(G)$ of a graph G is the minimum width of a tree decomposition of G.



Definition 1.2

A bramble *is a family of connected subgraphs of G that all touch each other: for every pair of disjoint subgraphs, there must exist an edge that has one endpoint in each subgraph. The* order *of a bramble is the smallest size of a set of vertices of G that has a nonempty intersection with each of the subgraphs.*

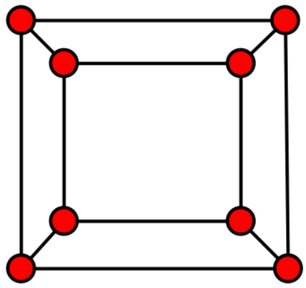


1.1 A few examples

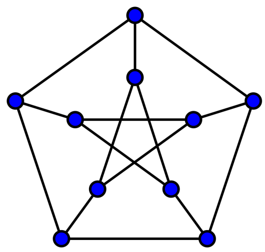
Theorem 1.3 (***Seymour, Thomas, 1993***)

A graph has a bramble of order k if and only if it has treewidth at least $k-1$.

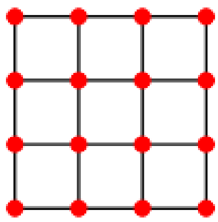
1. What is the treewidth of the cube graph?



2. What is the treewidth of the Petersen graph?



3. Let $n \geq 1$ be an integer. The grid graph, denoted G_n, is the graph with vertex set $[1, n]^2$ and an edge between vertices u and v if and only if $u - v \in \{(0,-1),(0,1),(-1,0),(1,0)\}$. What is the treewidth of G_n?



1.2 Upper-bounding the treewidth

1. Let Z be a set of vertices of G, and let B be the set of all subsets S of $V(G)$ such that $G[S]$ is connected and $|S \cap Z| > \frac{|Z|}{2}$.

- B is a bramble
- the order of the bramble B is the minimum size of a set $X \subseteq V(G)$ such that every component of $G - X$ contains at most half of the vertices of Z.

2. If every bramble in G has order at most k, then $tw(G) \leq 3k$.
hint 1: remember we proved the following last week:

- for a set Z of vertices of a graph G, let $G+Z$ denote the graph obtained from G by adding all edges between vertices of Z (turning it into a clique). Let X, Z be sets of vertices. For each component K of $G - X$, let $G_K = G[K \cup X]$ and $Z_K = (Z \cap K) \cup X$. If $tw(G_K + Z_K) \leq t$ for every component K of $G - X$, then $tw(G+Z) \leq \max(t, |Z \cup X| - 1)$.

hint 2: actually prove the following stronger claim: for every set Z of at most $2k+1$ vertices of G, the graph $G+Z$ has treewidth at most $3k$.