Combinatorics and Graph Theory III
Tutorial 6

Guillaume Aubian

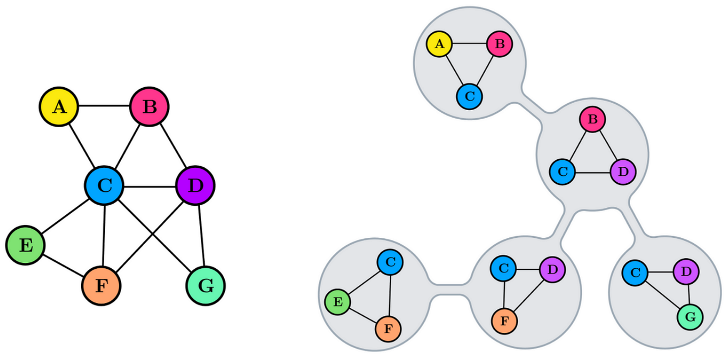
1 Treewidth

Definition 1.1

Given a graph G, a tree decomposition *is a pair (T, β) with T a tree and $\beta : V(T) \to 2^{V(G)}$. β assigns a bag $\beta(n)$ to each vertex n of T and obeys the following rules:*

- *for every $v \in V(G)$, there exists $n \in V(T)$ such that $v \in \beta(n)$,*
- *for every $uv \in E(G)$, there exists $n \in V(T)$ such that $u, v \in \beta(n)$,*
- *for every $v \in V(G)$, the set $\{n \in V(T) \mid v \in \beta(n)\}$ induces a connected subtree of T.*

The *width* of a tree decomposition is the size of the largest bag minus one. The *treewidth* $tw(G)$ of a graph G is the minimum width of a tree decomposition of G.



Prove the following:

1. Let (T, β) be a tree decomposition of a graph G. For a subtree T' of T, let $G[T']$ be the subgraph of G induced by $\bigcup_{x \in V(T')} \beta(x)$. If $e = uv$ is an edge of T and T' and T'' are the two components of $T - e$, then $G = G[T'] \cup G[T'']$ and $V(G[T'] \cap G[T'']) = \beta(u) \cap \beta(v)$ i.e. $X = \beta(u) \cap \beta(v)$ is a cut in G separating $G[T'] - X$ from $G[T''] - X$.

For an assignment w of non-negative weights to vertices of a graph G and a subgraph F of G, let $w(F)$ be the sum of the weights of the vertices of F.

2. If T is a tree, then there exists a vertex v such that every component K of $T - v$ satisfies $w(K) \leq \frac{w(T)}{2}$.
hint: a directed graph in which every vertex sees another vertex has a directed cycle.

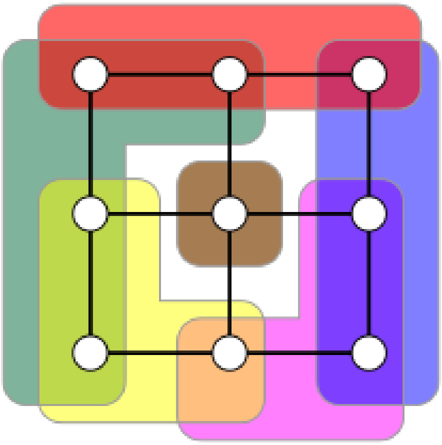
3. If G has treewidth k, then there exists a set X of at most $k + 1$ vertices of G such that every component K of $G - X$ satisfies $w(K) \leq \frac{w(G)}{2}$ i.e. graphs of small treewidth have small balanced cuts.

4. For a set Z of vertices of a graph G, let $G + Z$ denote the graph obtained from G by adding all edges between vertices of Z (turning it into a clique). Let X, Z be sets of vertices. For each component K of $G - X$, let $G_K = G[K \cup X]$ and $Z_K = (Z \cap K) \cup X$. If $tw(G_K + Z_K) \leq t$ for every component K of $G - X$, then $tw(G + Z) \leq \max(t, |Z \cup X| - 1)$.

2 Brambles

Definition 2.1

A bramble *is a family of connected subgraphs of G that all touch each other: for every pair of disjoint subgraphs, there must exist an edge that has one endpoint in each subgraph. The* order *of a bramble is the smallest size of a set of vertices of G that has a nonempty intersection with each of the subgraphs.*



A graph has a bramble of order k if and only if it has treewidth at least $k-1$.

1. Let Z be a set of vertices of G, and let B be the set of all subsets S of $V(G)$ such that $G[S]$ is connected and $|S \cap Z| > \frac{|Z|}{2}$.

- B is a bramble
- the order of the bramble B is the minimum size of a set $X \subseteq V(G)$ such that every component of $G - X$ contains at most half of the vertices of Z.

2. If every bramble in G has order at most k, then $tw(G) \leq 3k$.

hint 1: use the last two questions.

hint 2: actually prove the following stronger claim: for every set Z of at most $2k+1$ vertices of G, the graph $G + Z$ has treewidth at most $3k$.