Combinatorics and Graph Theory III
Tutorial 5

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1 Treewidth

Definition 1.1

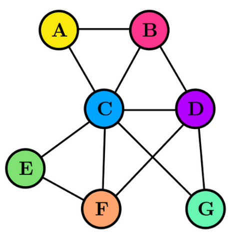
Given a graph G, a tree decomposition *is a pair (T, β) with T a tree and $\beta : V(T) \to 2^{V(G)}$. β assigns a bag $\beta(n)$ to each vertex n of T and obeys the following rules:*

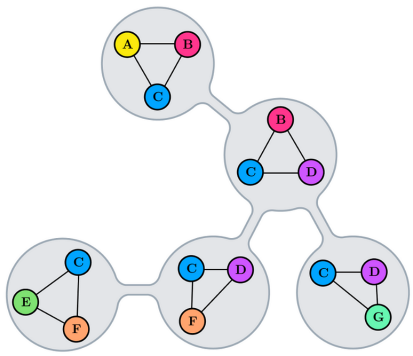
- *for every $v \in V(G)$, there exists $n \in V(T)$ such that $v \in \beta(n)$,*
- *for every $uv \in E(G)$, there exists $n \in V(T)$ such that $u, v \in \beta(n)$,*
- *for every $v \in V(G)$, the set $\{n \in V(T) \mid v \in \beta(n)\}$ induces a connected subtree of T.*

These rules can be summed up respectively as:

- every vertex is in some bag,
- every edge is in some bag,
- every vertex appears in a connected subtree of the decomposition.

The *width* of a tree decomposition is the size of the largest bag minus one. The *treewidth* $tw(G)$ of a graph G is the minimum width of a tree decomposition of G.





Prove the following:

1. paths have treewidth 1
2. trees have treewidth 1.
3. cycles have treewidth 2.
4. outerplanar graphs (planar graphs whose every vertex lie on a face) have treewidth 2.
5. graphs of treewidth t have a vertex of degree at most t.
6. the clique K_t has treewidth $t - 1$.
7. for any clique and tree decomposition of G there exists a bag which contains all vertices of the clique.
8. a clique-sum of two graphs of treewidth t have treewidth at most t.
9. if H is a minor of G, then $tw(H) \leq tw(G)$.