Combinatorics and Graph Theory III
Tutorial 4

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1 Hajós was wrong

Hajós conjectured the following :

Conjecture 1.1

For every $k \in \mathbb{N}$, every graph without K_k as a topological minor has chromatic number less than k.

This has been disproved since then. Let us see how!

Assume Hajós conjecture holds. Let $t \in \mathbb{N}$. Let G be a graph with n vertices which contains neither a clique nor an independent set of size t.

1. Prove that this implies that for $k = \lfloor \frac{n}{t} \rfloor$, G contains K_k as a topological minor.

For the following question, we need Turan's theorem (see proof on chalkboard):

Theorem 1.2

A graph on n vertex with no K_t has at most $(1 - \frac{1}{t})\frac{n^2}{2}$ edges.

2. Prove that this implies $|V(G)| \leq 2t^3$.

This is a contradiction since Erdös proved the following:

Theorem 1.3

For any $t \in \mathbb{N}$ there exist graphs on $2^{\lfloor \frac{t}{2} \rfloor}$ vertices with no clique nor independent set of size t.

3. The proof is nice and important: start from an empty graph on $n = \lfloor 2^{\frac{t}{2}} \rfloor$ vertices, add every possible edge with probability $\frac{1}{2}$ and upper-bounds the probability that there is a clique or an independent set on t vertices. Do the actual computation.

4. What is the smallest k for which this contradicts Hajós' conjecture?

2 2-Linkedness

Definition 2.1

G is 2-linked if for any four distinct vertices s_1, s_2, t_1, t_2, there exist vertex-disjoint s_it_i-path for $i = 1, 2$.

You can assume the following theorem:

Theorem 2.2

Every non-planar 4-connected graph is 2-linked.

1. What can you say about the minimum k such that every k-connected graph is 2-linked?

2. Let G be a plane 4-connected graph, and let s_1, s_2, t_1, t_2 be vertices of G not on the same face. Prove that G contains vertex-disjoint paths from s_1 to t_1 and from s_2 to t_2.

hint: Add a *well-chosen* cycle and argue that G becomes non-planar.