Combinatorics and Graph Theory III
Tutorial 2

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1 Clique-sums

Definition 1.1

Given two graphs G and H, such that vertices $g_1, g_2 \dots g_k \in V(G)$ form a clique in G and vertices $h_1, h_2 \dots h_k \in V(H)$ form a clique in H. The clique-sum of G and H relative to $g_1 \dots g_k$ and $h_1 \dots h_k$ is a graph obtained by identifying g_1 and h_1, g_2 and h_2 and so on, and then possibly deleting a subset of edges from that clique.

Informally, we glue G and H along cliques of the same size, and remove edges from that clique.

1. Prove that if K_n is a minor of a clique-sum of two graphs G_1 and G_2, it is either a minor of G_1 or G_2.
2. Prove that if two graphs G_1 and G_2 are k-colourable, so is any clique-sum of G_1 and G_2.

2 Classes defined by forbidding minors

Definition 2.1

A vertex cover is a set of a set of vertices such that each edge is adjacent to one of these vertices.

1. What are the graphs that do not contain $2K_2$ as a minor?
2. Let $t \in \mathbb{N}$. Prove that if a graph has a vertex cover of size t, so do all its minors.

This means that graphs with vertex cover at most t can be defined as the graphs that do not contain some obstructions as minors.

3. What are the minimal forbidden minors for having a vertex cover of size at most 1?
hint: you should find 2 such graphs!
4. What are the minimal forbidden minors for having a vertex cover of size at most 2?
hint: you should find 4 such graphs!
5. Let $t \in \mathbb{N}$. Prove that a graph with vertex cover of size t does not contain $(t+1)K_2$ as a subgraph.
6. Let $t \in \mathbb{N}$. Prove that a graph without $(t+1)K_2$ as a subgraph has a vertex cover of size at most $2t$.

3 A slightly harder problem

Let $t \in \mathbb{N}$. Let G be a graph that does not contain $K_{1,t}$ as a minor. Prove that all but at most $10t$ vertices of G have degree at most two.