Combinatorics and Graph Theory III
Tutorial 1

Guillaume Aubian

1 Bipartite graphs

Definition 1.1

Let $k \in \mathbb{N}$. A graph G is k-partite if you can partition its vertices into k disjoint independent sets.

Determining whether a graph is k-partite is hard, but in the case $k = 2$, there is a simple characterization:

Theorem 1.2

A graph is bipartite if and only if it has no odd induced cycle.

Two remarks:

- a graph has an odd cycle if and only if it has an odd induced cycle,
- a closed walk is a cycle allowed to reuse an edge/a vertex multiple times. A graph has an odd cycle if and only if it has an odd closed walk.

1. Prove that if a graph is bipartite, then it has no odd induced cycle.
2. Prove that if a graph has no odd induced cycle, then it is bipartite.

hint: If a graph is bipartite, fix a vertex v: there is a simple characterization of each part of the bipartition.

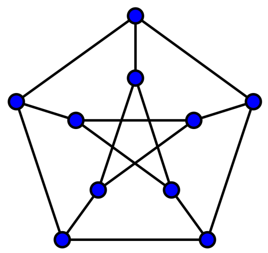
2 Minors

A graph H is a minor of a graph G if it can be obtained from G by repeated applications of the following rules:

1. delete a vertex
2. delete an edge
3. contract an edge

If only rules (1) and (2) are used, H is a subgraph of G. If only rule (1) is used, H is an induced subgraph of G. If only rule (3) is applied and only whenever one endpoint of the edge has degree 2, H is a subdivision of G.

1. Prove that we can always delete vertices, then delete edges, then contract edges.
2. Prove that forests are exactly the graphs that do not contain C_3 as a minor.
3. Is $K_{3,3}$ a minor of the Petersen graph ?



3 Kuratowski's theorem

A graph is planar if it admits a planar embedding. A planar embedding is a is a drawing of a graph in which edges do not cross outside of vertices. But it is cumbersome to define it properly:

Definition 3.1

A planar embedding of a graph $G = (V, E)$ is an injective function $\psi : V \to \mathbb{R}^2$ and a set of continuous functions $\varphi_{uv} : [0, 1] \to R$ such that:
- $\varphi_{uv}(0) = \psi(u)$,
- $\varphi_{uv}(1) = \psi(v)$,
- $\forall t, t' \in]0, 1[, \forall e, e' \in E, \varphi_e(t) = \varphi'_e(t')$ *if and only if* $e = e'$ *and* $t = t'$.

Fortunately, there exists a nice structural characterization of planar graphs:

Theorem 3.2 (*Kuratowski's theorem*)

A graph is planar if and only if it does not contain any subdivision of $K_{3,3}$ or K_5.

This also exists in a flavour using minors instead of subdivisions:

Theorem 3.3 (*Wagner's theorem*)

A graph is planar if and only if it does not contain $K_{3,3}$ or K_5 as a minor.

1. Using Euler's formula, prove that $K_{3,3}$ and K_5 are not minors.
2. Argue that if a graph is planar, so are all its minors.
3. Prove that Kuratowski's theorem implies Wagner's theorem.
4. Prove that containing $K_{3,3}$ as a minor is equivalent to having it as a subdivision.
5. Prove that containing K_5 as a minor implies containing K_5 or $K_{3,3}$ as a subdivision.

4 Surfaces

A surface is a two-dimensional manifold, that is a two-dimensional object that locally looks like the Euclidean plane. Some examples of surfaces: the plane, the projective plane, the sphere, the torus, the Klein bottle.

1. Argue that a graph can be embedded on a sphere if and only if it can be embedded in the plane.
2. does Pac-Man live on a sphere?
3. draw an embedding of K_5 on a torus.
4. argue that every minor of a graph which can be embedded on a surface can be embedded in this surface.

5 Chordal Graphs

Definition 5.1

A graph is chordal if its only induced cycles have length 3.

An important fact about chordal graphs is that they contain a simplicial vertex:

Theorem 5.2

Every chordal graph contains a vertex whose neighbourhood is a clique. Such a vertex is called simplicial.

1. What can we say about the induced subgraphs of a chordal graph?

Thus, we can always *peel off* a chordal graph by repeatedly removing simplicial vertices. But does the reverse process always produce a chordal graph?

2. Let G be a chordal graph, and K a clique of G. Add to a G a vertex that sees exactly K. Prove that the obtained graph is chordal.

This proves that chordal graphs are exactly those that can be obtained by repeatedly adding simplicial vertices.