Combinatorics and Graph Theory III
Homework

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Hand out your answers by November 30 either at guillaume.aubian@gaubian.xyz or during the tutorial.

A few reminders: H is a *subgraph* of G if H can be obtained from G by successive deletions of vertices and edges. If only vertices are allowed to be deleted, we say that H is an *induced subgraph* of G. If H is a subgraph of G and $H \neq G$, we say that H is a *proper subgraph* of G. A *colouring* of a graph is an assignment of colours to its vertices so that no two adjacent vertices receive the same colour. The *chromatic number* of a graph is the least number of colours needed in a colouring of this graph.

Let $k \in \mathbb{N}$. A graph is said to be *k-critical* if it has chromatic number k and all its proper subgraphs have chromatic number at most $k-1$.

1. Prove that in a $(k+1)$-critical graph, every vertex has degree at least k.

2. Prove that every graph of chromatic number at least k contains (as a subgraph) a k-critical subgraph.

3. Prove that if G is a $(k+1)$-critical graph, G contains (as a subgraph) every forest on k vertices.
hint: prove this either by contradiction or by induction on k.

This implies that for every forest F, graphs not containing F as a subgraph have bounded chromatic number. This raises the question of whether there exists any non-forest with this property. In order to study this, let us use the following theorem by Erdös:

Theorem 1 (*Erdös, 1959*)

Let $\ell, k \in \mathbb{N}$. There exists a graph with no cycle on less than ℓ vertices and chromatic number at least k.

4. Let G be a graph such that graphs not containing G as a subgraph have bounded chromatic number. Deduce from this theorem that G is a forest.

This raises the question of whether a corresponding result exists for induced subgraphs.

5. Let G be a graph such that graphs not containing G as an **induced** subgraph have bounded chromatic number. Prove that G is either the empty graph, a single vertex or a single edge.
hint: prove that G is both a forest and a complete graph.

Similarly, one could prove that if a finite set X of graphs is such that there exist graphs of arbitrarily large chromatic number not containing any graph of X as an induced subgraph, then X contains a complete graph and a forest. Whether the converse is true is an open problem: the Gyárfás-Sumner conjecture.

Conjecture 2 (*Gyáfás, 1975 & Sumner, 1981*)

Let F be a forest and K a clique. Graphs not containing either F nor K have bounded chromatic number.